# MATB44 Week 1 Notes

1. Introduction:

- Differential equations of the first order can be written as $\frac{dy}{dt} = f(y,t)$.
- However, there is no general method for solving the eqn in terms of elementary functions. Instead, we'll describe several methods, each of which is applicable to a certain subclass of first-order equations.

2. Linear Differential Eqns/Method of Integrating Factors:

- We usually write the general first-order linear differential eqn in the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

where p and g are given functions of the independent variable t. However, sometimes it is more convenient to write the eqn in the form

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

where P, Q and G are given.
If $P(t) \neq 0$, then we can convert

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \quad (1)$$

to

$$\frac{dy}{dt} + p(t)y = g(t) \quad (2)$$

by dividing both sides of eqn (1) by $P(t)$.

- Note: We know that eqns (1) and (2) are of order 1 because they have $\frac{dy}{dx}$.

- Note: $P(t)$, $Q(t)$, $G(t)$, $p(t)$, $g(t)$ are known while $y$ is unknown.

- If we are given an eqn that looks like eqn (1), we can sometimes assume that
$$P(t)\frac{dy}{dt} + Q(t)y = \frac{d}{dt}\left(f(t)y\right)$$
for some function f which we know.
Then:
  a) $(f(t)y)' = G(t)$
  b) $f(t)y = \int G$

- E.g. 1 Solve $t^2y' + 2ty = t^3$

Soln:

Here,

$P(t) = t^2$

$Q(t) = 2t$

$G(t) = t^3$

Looking at $t^2y' + 2ty$, we see that it is equal to $(t^2y)'$.

$$\begin{aligned}(t^2y)' &= (t^2)'y + (t^2)(y') \\ &= 2ty + t^2y' \\ &= t^2y' + 2ty\end{aligned}$$

Hence, we can rewrite $t^2y' + 2ty = t^3$ as $(t^2y)' = t^3$. (This is $(f(t)y)' = G(t)$).

$$\begin{aligned}t^2y &= \int t^3\,dt \\ &= \frac{t^4}{4} + c\end{aligned}$$

$$y = \frac{1}{t^2}\left(\frac{t^4}{4} + c\right)$$

- Eg. 2 Solve $(4+t^2)y' + 2ty = 4t$

Soln:
$(4+t^2)y' + 2ty = ((t^2+4)y)'$

Hence, we can rewrite the original equation as $((t^2+4)y)' = 4t$.

$$(t^2+4)y = \int 4t\,dt$$
$$= 2t^2 + c$$
$$y = \frac{2t^2+c}{t^2+4}$$

Solns to

- Note: First order differential eqns always have one numerical parameter which varies and is sometimes even arbitrary. I.e. There is 1 constant.

However, solns to second order differential eqns always have 2 constants.

The constant governs "the initial conditions".

Eg. 3 Solve $t^2y' + 2ty = t^3$ where $y(1) = \frac{3}{4}$.

Soln:

Earlier, we got $y = \frac{1}{t^2}\left(\frac{t^4}{4} + c\right)$.

$y(1) = \frac{3}{4}$ means that when $t=1$, $y = \frac{3}{4}$.

Plug in 1 for t and $\frac{3}{4}$ for y and we

get $\frac{3}{4} = \frac{1}{1^2}\left(\frac{1^4}{4} + c\right)$

$$= \frac{1}{4} + c$$

$$c = \frac{1}{2}. \qquad y = \frac{1}{t^2}\left(\frac{t^4}{4} + \frac{1}{2}\right)$$

For this initial condition, $c = \frac{1}{2}$.

- Note: Examples 1 and 2 are special cases. Sometimes, the eqn isn't always of the form $P(t)\frac{dy}{dt} + Q(t)y = G(t)$.

- If we're given eqns of the form $\frac{dy}{dt} + p(t)y = g(t)$, then we have to use the method of integrating factors.

- Here are the steps:
  1. Start with this form:

$$\frac{dy}{dt} + p(t)y = g(t)$$

  2. Multiply the above eqn by $\mu(t)$.
     $\mu(t)$ is called an integrating factor.
     Now we have

$$(\mu(t))\left(\frac{dy}{dt}\right) + (\mu(t))(p(t))y = (\mu(t))(g(t))$$

  3. Assume that

$$(\mu(t))\left(\frac{dy}{dt}\right) + (\mu(t))(p(t))y = (\mu(t)y)'$$

Expanding the eqn above, we get

$$(\mu(t))\left(\frac{dy}{dt}\right) + (\mu(t))(p(t))y = (\mu(t)y') + ((\mu(t))'y)$$

Since $(\mu(t))\left(\frac{dy}{dt}\right) = (\mu(t))(y')$,

we can cancel them out. Now, we're left with

$$(\mu(t))(p(t))y = (\mu(t))'y$$

$$(\mu(t))\,p(t) = (\mu(t))'$$

$$p(t) = \frac{(\mu(t))'}{\mu(t)}$$

Recall that $\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$.

Hence, $\frac{(\mu(t))'}{\mu(t)} = \frac{d}{dt}\left(\ln(\mu(t))\right)$

$$p(t) = \frac{d}{dt}\left(\ln(\mu(t))\right)$$

$$\int p(t)\,dt = \ln(\mu(t))$$

I integrated both sides.

$$e^{\int p(t)\,dt} = \mu(t)$$

I raised both sides by e.
$e^{\ln x} = x$

4. Since we assumed that
$\left(\mu(t)\right)\left(\frac{dy}{dt}\right) + \left(\mu(t)\right)\left(p(t)\right)y = \left(\mu(t)y\right)'$,
we can substitute $(\mu(t)y)'$ in our eqn. Now, we have:

$$(\mu(t)y)' = (\mu(t))(g(t))$$

Integrating both sides, we get

$$\int (\mu(t)y)'\,dt = \int (\mu(t))(g(t))\,dt$$

$$\mu(t)y = \int (\mu(t))(g(t))\,dt$$

$$y = \frac{\int (\mu(t))(g(t))\,dt}{(\mu(t))}$$

5. Now, we have:

a) $\mu(t) = e^{\int p(t)\,dt}$

b) $y = \dfrac{\int (\mu(t))(g(t))\,dt}{(\mu(t))}$

– E.g. 4 Solve $y' + 2ty = 4t$

Solns:

1. Multiply both sides of the eqn by $\mu(t)$.
$(\mu(t))y' + (2)(\mu(t))(t)(y) = (4)(\mu(t))(t)$

2. Assume that the LHS of the new equation equals to $(\mu(t)y)'$.

$$\cancel{(\mu(t))y'} + 2(\mu(t))(t)(y) = (\mu(t)y)'$$
$$= (\mu(t))'y + \cancel{\mu(t)y'}$$
$$2(\mu(t))(t)\cancel{(y)} = (\mu(t))'\cancel{y}$$
$$2t(\mu(t)) = (\mu(t))'$$
$$2t = \frac{(\mu(t))'}{(\mu(t))}$$
$$= (\ln(\mu(t)))'$$

$$\int 2t\,dt = \ln(\mu(t))$$
$$t^2 + c = \ln(\mu(t))$$

$$e^{t^2+c} = \mu(t)$$
$$e^{t^2} \cdot e^c = \mu(t)$$ Recall that $e^{a+b} = e^a \cdot e^b$

I will use c' to represent $e^c$.
$$e^{t^2} \cdot c' = \mu(t)$$

3. Substitute $(\mu(t)y)'$ for the LHS.

$$(\mu(t)y)' = 4(\mu(t))(t)$$
Recall that $\mu(t) = e^{t^2} \cdot c'$.
We will set $c' = 1$.
$$(e^{t^2}y)' = 4(e^{t^2})(t)$$
$$e^{t^2} \cdot y = \int 4(e^{t^2})(t)\,dt$$
$$= 4\int (e^{t^2})(t)\,dt$$

To solve $4\int(e^{t^2})(t)\,dt$, we will do integration by substitution.

Let $u = t^2$
$$\frac{du}{dt} = 2t$$
$$dt = \frac{du}{2t}$$

Now, we have $\frac{4}{2}\int e^u\,du$

$$= 2\int e^u\,du$$
$$= 2e^u + C$$

$$= 2e^{t^2} + c$$

$$e^{t^2} \cdot y = 2e^{t^2} + c$$
$$y = \frac{2e^{t^2} + c}{e^{t^2}}$$
$$= 2 + c \cdot e^{-t^2}$$

- E.g. 5. Solve $y' + \frac{y}{2} = \frac{e^{t/3}}{2}$

Soln:

1. $(\mu(t))y' + \frac{(\mu(t))y}{2} = \frac{(\mu(t))e^{t/3}}{2}$

2. $\cancel{(\mu(t))y'} + \frac{(\mu(t))y}{2} = (\mu(t)y)'$

$$= (\mu(t))'y + \cancel{(\mu(t))y'}$$

$$\frac{(\mu(t))\cancel{y}}{2} = (\mu(t))'\cancel{y}$$
$$\frac{1}{2} = \frac{(\mu(t))'}{(\mu(t))}$$
$$= (\ln(\mu(t)))'$$
$$\int \frac{1}{2}\,dt = \ln(\mu(t))$$
$$\frac{t}{2} + c = \ln(\mu(t))$$

$$e^{\frac{t}{2} + c} = \mu(t)$$
Let $c' = e^c$
$$e^{t/2} \cdot c' = \mu(t)$$

3. $(\mu(t)y)' = \frac{(\mu(t))e^{t/3}}{2}$

let $c' = 1$

$(e^{t/2} \cdot y)' = \frac{(e^{t/2})(e^{t/3})}{2}$

$= \frac{e^{5t/6}}{2}$

$e^{t/2} \cdot y = \int \frac{e^{5t/6}}{2}\, dt$

$= \frac{1}{2} \int e^{5t/6}\, dt$

To solve $\frac{\int e^{5t/6}\, dt}{2}$, we will do integration by substitution.

let $u = \frac{5t}{6}$

$\frac{du}{dt} = \frac{5}{6}$

$dt = \frac{6\,du}{5}$

$\frac{\int e^{5t/6}}{2} dt$ becomes $\frac{6}{10} \int e^{u}\, du$

$= \frac{3}{5} e^{u} + c$

$= \frac{3}{5} e^{5t/6} + c$

$$e^{t/2} \cdot y = \frac{3}{5} e^{\frac{5t}{6}} + c$$

$$y = \frac{1}{e^{t/2}} \left( \frac{3}{5} e^{\frac{5t}{6}} + c \right)$$

$$= \frac{3}{5} e^{t/3} + c \cdot e^{-t/2}$$

## 3. Separable Equations:

- A separable differential equation is any differential equation that can be written as:

  a) $M(x) + N(y) \dfrac{dy}{dx} = 0$

  OR

  b) $M(x)\,dx + N(y)\,dy = 0$

- Note: $Y$ is a function of $x$. I.e. $y = y(x)$

- To solve this differential eqn, we first
  move all the $Y$'s to one side of the eqn and move all the $x$'s to the other side.

  I.e. $N(y)\,dy = M(x)\,dx$

Next, we integrate both sides.

I.e. $\int N(y)\,dy = \int M(x)\,dx$

- E.g. 6. Solve $y' = \dfrac{x^2}{y(1+x^3)}$

Soln:

$$y \cdot y' = \frac{x^2}{1+x^3}$$

$$y\frac{dy}{dx} = \frac{x^2}{1+x^3}$$

$$y\,dy = \frac{x^2}{1+x^3}\,dx$$

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx$$

$$\int y\,dy = \frac{y^2}{2}$$

$$\int \frac{x^2}{1+x^3}\,dx$$

Let $u = 1+x^3$

$$\frac{du}{dx} = 3x^2$$

$$dx = \frac{du}{3x^2}$$

$$\frac{1}{3}\int \frac{1}{u}\,du$$

$$\frac{1}{3}\left(\ln|u| + c\right)$$

$$\frac{1}{3}\left(\ln|1+x^3| + c\right)$$

$$\frac{y^2}{2} = \frac{1}{3}\left(\ln|1+x^3|\right) + c$$

$$\underbrace{\frac{y^2}{2} - \frac{\ln|1+x^3|}{3}}_{x \text{ and } y \text{ on LHS}} = \underbrace{c}_{\text{Constant on RHS}}$$

- E.g. 7 Solve $y' + y^2\sin(x) = 0$

Soln:

$$\frac{dy}{dx} + y^2\sin(x) = 0$$

$$\frac{dy}{dx} = -y^2\sin(x)$$

$$\frac{1}{y^2}\,dy = -\sin(x)\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -\sin(x)\,dx$$

$$\frac{-1}{y} = \cos(x) + c$$

$$\underbrace{\frac{-1}{y} - \cos(x)}_{x \text{ and } y \text{ on LHS}} = \underbrace{c}_{\text{constant on RHS}}$$

## 4. More Examples:

a). Solve $ty' + 2y = 4t^2$ s.t. $y(1) = 2$

Soln:

First, we need to change the eqn to this form: $\frac{dy}{dt} + p(t)y = g(t)$.

To do so, we will divide both sides of the eqn by $t$.

Hence, we get: $y' + \frac{2y}{t} = 4t$

Next, we will multiply both sides of the new equation by $\mu(t)$.

$$\mu(t)y' + \frac{\mu(t)2y}{t} = 4t\mu(t)$$

Assume that the LHS $= (\mu(t)y)'$

$$\cancel{\mu(t)y'} + \frac{\mu(t)2y}{t} = (\mu(t)y)'$$

$$= (\mu(t))'y + \cancel{\mu(t)y'}$$

$$\frac{\mu(t)2\cancel{y}}{t} = (\mu(t))'\cancel{y}$$

$$\frac{2}{t} = \frac{(\mu(t))'}{\mu(t)}$$

$$= (\ln(\mu(t)))'$$

$$\int \frac{2}{t}\,dt = \ln(\mu(t))$$

$$2\ln|t| + c = \ln(\mu(t))$$

$$e^{2\ln|t|+c} = \mu(t)$$

$$e^{2\ln|t|} \cdot e^{c} = \mu(t)$$

let $c' = e^c$

$$e^{2\ln|t|} \cdot c' = \mu(t)$$

$$\underbrace{\left(e^{\ln|t|}\right)^2}\cdot c' = \mu(t)$$

Recall that $a^{2x} = (a^x)^2$

$$t^2 \cdot c' = \mu(t)$$

Recall that $a^{\log_a x} = x$ and
$\ln x = \log_e x$.

$$\mu(t) = t^2 \cdot c'$$

We can set $c'$ to 1.

$$\mu(t) = t^2$$

Next, we will replace $\mu(t)y' + \dfrac{\mu(t)2y}{t}$ with $(\mu(t)y)'$ and solve $(\mu(t)y)' = 4t(\mu(t))$.

$$(\mu(t)y)' = 4t(\mu(t))$$
$$\mu(t)y = \int 4t(\mu(t))\,dt$$
$$yt^2 = \int 4t(t^2)\,dt$$
$$= \int 4t^3\,dt$$
$$= t^4 + c$$
$$y = \frac{t^4+c}{t^2}$$
$$= t^2 + \frac{c}{t^2}$$

Now we plug 1 for t and 2 for y.

$$2 = 1^2 + \frac{c}{1^2}$$
$$= 1 + c$$
$$c = 1$$

$$y = t^2 + \frac{1}{t^2}$$

b) Solve $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

Soln:

1. Divide both sides of the eqn by $(1+t^2)$.

$$y' + \frac{4ty}{1+t^2} = (1+t^2)^{-3} \qquad (1)$$

2. Multiply both sides of eqn 1 by the integrating factor, $\mu(t)$.

$$\mu(t)y' + \frac{(4ty)(\mu(t))}{1+t^2} = \frac{\mu(t)}{(1+t^2)^3} \qquad (2)$$

3. Assume LHS of eqn 2 $= (\mu(t)y)'$

$$\mu(t)y' + \frac{(4ty)(\mu(t))}{1+t^2} = (\mu(t)y)'$$

$$= (\mu(t))'y + \mu(t)y' \qquad (3)$$

4. Simply eqn 3 and solve for $\mu(t)$

$$\cancel{\mu(t)y'} + \frac{(4t\cancel{y})(\mu(t))}{1+t^2} = (\mu(t))'\cancel{y} + \cancel{\mu(t)y'}$$

$$\frac{(4t)(\mu(t))}{1+t^2} = (\mu(t))'$$

$$\frac{4t}{1+t^2} = \frac{(\mu(t))'}{(\mu(t))}$$

$$= (\ln(\mu(t)))'$$

$$\int \frac{4t}{1+t^2}\, dt = \ln(\mu(t))$$

To solve $\int \frac{4t}{1+t^2}\, dt$, I will use integration by substitution.

Let $u = 1+t^2$

$\frac{du}{dt} = 2t$

$dt = \frac{du}{2t}$

$$\int \frac{4t}{1+t^2}\, dt \longrightarrow \int \frac{2}{u}\, du$$

$$= 2\int \frac{1}{u}\, du$$

$$= 2 \ln |u| + c$$

$$= 2 \ln |1+t^2| + c$$

$$2\ln(1+t^2)+c = \ln(\mu(t))$$

$$e^{2\ln(1+t^2)+c} = \mu(t)$$

let $c' = e^c$

$$e^{2\ln(1+t^2)} \cdot \underline{e^c} = \mu(t)$$

Recall that $e^{a+b} = e^a \cdot e^b$

$$\underline{(e^{\ln(1+t^2)})^2} \cdot e^c = \mu(t)$$

Recall that $e^{ax} = (e^x)^a$

$$\underline{(1+t^2)^2} \cdot c' = \mu(t)$$

Recall that:

a) $a^{\log_a b} = b$

b) $\ln x = \log_e x$

Set $c'$ to 1.

$\mu(t) = (1+t^2)^2$ (4)

5. Replace the LHS of eqn 2 with $(\mu(t)y)'$ and solve for y.

$$(\mu(t)y)' = \frac{\mu(t)}{(1+t^2)^3}$$

$$(\mu(t))y = \int \frac{\mu(t)}{(1+t^2)^3}\,dt$$

$$(1+t^2)^2 y = \int \frac{(1+t^2)^2}{(1+t^2)^3}\,dt$$

$$= \int \frac{1}{1+t^2}\,dt$$

$$(1+t^2)^2 y = \arctan(t) + c$$
$$y = \frac{\arctan(t)+c}{(1+t^2)^2}$$

c) Solve $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

Soln:

$1-y^2\, dy = x^2\, dx$

$\int 1-y^2\, dy = \int x^2\, dx$

$y - \frac{y^3}{3} = \frac{x^3}{3} + c$

$\underbrace{y - \frac{y^3}{3} - \frac{x^3}{3}}_{x \text{ and } y \text{ on LHS}} = \underbrace{c}_{\text{Constant on RHS}}$ } This is called the General Solution.

d) Solve $\frac{dy}{dx} = \frac{2x}{1+2y}$, $y(2) = 0$

Soln:

$1+2y\, dy = 2x\, dx$

$\int 1+2y\, dy = \int 2x\, dx$

$y + y^2 = x^2 + c$

Plug in 2 for x and 0 for y

$0 = 4 + c$

$c = -4$

$$y + y^2 = x^2 - 4$$